Linear systems – Final exam

Final exam 2018–2019, Tuesday 18 June 2019, 9:00 – 12:00

Instructions

1. The use of books, lecture notes, or (your own) notes is not allowed.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1 (10 points)

Solve the initial value problem

$$t\dot{x}(t) - x(t) = t^3 \sin(t^2), \qquad x(\sqrt{\pi}) = 0.$$

Problem 2 ($3 + 3 + 6 = 12$ points)

Consider the scalar differential equation

$$\frac{\mathrm{d}^3 q}{\mathrm{d}t^3}(t) + q^2(t)\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t) + q(t) - q^2(t) = u(t). \tag{1}$$

(a) Write the differential equation (1) in state-space form $\dot{x}(t) = f(x(t), u(t))$ by using the state $x_1(t) = q(t)$, $x_2(t) = \dot{q}(t)$, $x_3(t) = \ddot{q}(t)$.

(b) Show that, for $u(t) = \bar{u} = 0$ for all $t \geq 0$,

$$\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

is an equilibrium point. In addition, compute all other equilibria (again for $\bar{u} = 0$).

(c) Determine the linearization of (1) around the equilibrium (2).

Problem 3 (16 points)

Consider the linear system given by the transfer function

$$T(s) = \frac{s}{s^4 + as^3 + as^2 + as + 1},$$

where $a \in \mathbb{R}$. Determine all values of a for which the system is externally stable.

Problem 4

(4 + 12 + 8 = 24 points)

Consider the linear system

$$\mathbf{\Sigma}: \quad \dot{x}(t) = Ax(t) + Bu(t),$$

with state $x(t) \in \mathbb{R}^3$, input $u(t) \in \mathbb{R}$, and

$$A = \begin{bmatrix} -1 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -5 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Is the system $\mathbf{\Sigma}$ controllable?

(b) Find a nonsingular matrix T such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

for some matrices A_{11}, A_{12}, A_{22}, and B_1 with the matrix pair (A_{11}, B_1) controllable. In addition, compute the eigenvalues of A_{22}.

(c) Using the matrix T obtained in (b), find a state feedback $u(t) = Fx(t)$ that stabilizes $\mathbf{\Sigma}$.

Problem 5

(4 + 6 + 2 = 12 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -8 & -10 & 0 \\ 5 & 7 & 0 \\ -6 & -10 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} x(t).$$

(a) Is the system observable? If not, find a basis for the unobservable subspace.

(b) Is the system detectable?

(c) Does there exist a stable state observer for the system?

Problem 6

(16 points)

Show that, for any matrix G, the matrix pair $(A - GC, C)$ is observable if and only if the matrix pair (A, C) is observable.